Storgruppsövning 7/11-13

Functional equations

$f(s+t) = f(s) + f(t) \quad \forall\, s,t \geq 0 \quad \Rightarrow f(t) = c\,t$
$g(s+t) = g(s)g(t) \quad \forall\, s,t > 0 \quad \Rightarrow g(t) = \exp(c\,t)$
$f = \log(g).$

$f(s+t) = f(s) + f(t) \Rightarrow f(t) = c\,t$
Proof: $f_t'(s+t) = 0 + f'(t)$
$t = 0 \Rightarrow f'(s) = f'(0)$

$$\int_0^t f(s+x)\,dx = \int_0^t f(s)\,dx + \int_0^t f(x)\,dx$$

$$F(s+t) - F(s) = t f(s) + F(t) - F(0)$$

5.83
$X_n = Z_1 + \dots + Z_n$ $(X_n, n \geq 1)$, $Z_1, Z_2, \dots$ IID r.v's with zero mean, variance σ^2.
Is X_n stationary?
Solution:

$$Var(X_n) = Var\left(\sum_{i=1}^n Z_i\right) = \sum_{i=1}^n Var(Z_i) = n\sigma^2$$

the variance is not the same, that means it's not stationary.

$$\left(f_{X_n}(x) = f_{Z_1+\dots+Z_n}(x)\right)$$

$$\psi_X(w) = E(e^{iwX})$$

$$\psi_{X_n}(w) = E(e^{iwX_n}) = E(e^{iw(Z_1+\dots+Z_n)}) = E(e^{iwZ_1} \cdot \ldots \cdot e^{iwZ_n}) = \{E(XY) = E(X)E(Y)\}$$

$$= E(e^{iwZ_1}) \cdot \ldots \cdot E(e^{iwZ_n}) = (E(e^{iwZ_1}))^n$$

Answer: No, it's not stationary.

5.84
$X(t) = Y\cos(wt + \Theta)$ for $t \in \mathbb{R}$ where Y and Θ are independent r.v's uniformly distributed over $(-A, A)$ and $(-\pi, \pi)$ respectively. Find $\underbrace{\mu_X(t)}_{E(X(t))}$ and $\underbrace{R_X(s,t)}_{E(X(s)X(t))}$.

Solution:

$$E(X(t)) = E(Y\cos(wt+\Theta)) = \underbrace{E(Y)}_{=0}\underbrace{\underbrace{E(\cos(wt+\Theta))}_{=0}}_{=\int_{-\pi}^{\pi}\cos(wt+\theta)\underbrace{\frac{1}{2\pi}}_{f_\Theta(\theta)}d\theta} = 0$$

$$R_X(s,t) = E(X(s)X(t)) = E(Y^2\cos(ws+\theta)\cos(wt+\theta)) =$$

$$= E(Y^2)E(\cos(ws+\theta)\cos(wt+\theta)) = \Big\{ \tfrac{1}{2}(e^{ix}+e^{-ix})(e^{iy}+e^{-iy}) =$$

$$= \tfrac{1}{4}(e^{i(x+y)} + e^{-i(x+y)}) + \tfrac{1}{4}(e^{i(x-y)} + e^{-i(x-y)}) = \tfrac{1}{2}\cos(x+y) + \tfrac{1}{2}\cos(x-y) \Big\} =$$

$$= \int_{-A}^{A} y^2 \underbrace{\tfrac{1}{2A}}_{f_Y(y)} dy \cdot \Big[\underbrace{E(\tfrac{1}{2}\cos(w(s+t)+2\theta))}_{=0} + \underbrace{E(\tfrac{1}{2}\cos(w(s-t)))}_{\text{doesn't have r.v's}}\Big] = \frac{A^2}{3} \cdot \frac{1}{2}\cos(w(s-t))$$

Answer: $\mu_X(t) = 0$ and $R_X(s,t) = \frac{A^2}{3} \cdot \frac{1}{2}\cos(w(s-t))$

5.85

$X(t)$ WSS $R_X(t, t+T) = e^{-|T|/2}$

Find $E(X(5)^2)$ and $E((X(5)-X(3))^2)$.

Solution:

$R_X(t, t+T) = E(X(t)X(t+T))$

$E(X(5)^2) = \{T=0, t=5\} = e^{-|0|/2} = 1$

$$E((X(5)-X(3))^2) = \underbrace{E(X(5)^2)}_{=1} - \underbrace{2E(X(3)X(5))}_{t=3,\,T=2 \Rightarrow e^{-1}} + \underbrace{E(X(3)^2)}_{=1} = 2(1-e^{-1})$$

Answer: $E(X(5)^2) = 1$, $E((X(5)-X(3))^2) = 2(1-e^{-1})$.

5.86

$X(t) = U\cos(t) + (V+1)\sin(t)$, $t \in \mathbb{R}$

U and V independent r.v's, $E(U) = E(V) = 0$, $E(U^2) = E(V^2) = 1$

Find $K_X(s,t) = \mathrm{Cov}(X(s), X(t))$, is $X(t)$ WSS?

Solution:

$$\mu_X(t) = E(X(t)) = E(U\cos(t) + (V+1)\sin(t)) = \cos(t)\underbrace{E(U)}_{=0} + \sin t\, \underbrace{E(V+1)}_{=0} = \sin(t)$$

$$\Big\{E\Big(\sum_{i=1}^{n} a_i X_i\Big) = \sum_{i=1}^{n} a_i E(X_i)\Big\}$$

$$K_X(s,t) = \mathrm{Cov}(U\cos(s) + (V+1)\sin(s),\ U\cos(t) + (V+1)\sin(t)) =$$

$$= \Big\{\mathrm{Cov}\Big(\sum_{i=1}^{m} a_i X_i,\ \sum_{j=1}^{n} b_j Y_j\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{Cov}(X_i, Y_j)\Big\} =$$

$$= \underbrace{\mathrm{Cov}(U,U)}_{=(*)}\cos(s)\cos(t) + \underbrace{\mathrm{Cov}(U,V+1)}_{=0}\cos(s)\sin(t) + \underbrace{\mathrm{Cov}(V+1,U)}_{=0}\sin(s)\cos(t) +$$

$$+ \underbrace{\mathrm{Cov}(V+1,V+1)}_{\mathrm{Var}(V+1)=\mathrm{Var}(V)=1}\sin(s)\sin(t) = \cos(s-t)$$

$(*) = E((U-E(U))^2) = E(U^2) = 1$

Answer: It's not WSS, because $\mu_X(t) = \sin(t) \neq$ const.
$K_X(s,t) = \cos(s-t)$.

Computational problem 1

$\Delta_\varepsilon(t) = \frac{W(t-\varepsilon) - W(t)}{\varepsilon}$ for $t>0$, for a small $\varepsilon>0$

where $(W(t), t>0)$ is the Wiener process, i.e., a stationary independent process with $W(t)-W(s)$ $N(0, t-s)$-distributed
$\Leftrightarrow$ $(W(t), t>0)$ normal process with zero-mean and
$R_X(s,t) = K_X(s,t) = \min(s,t)$

Show $R_{\Delta_\varepsilon}(t, t+T) =$

Solution:

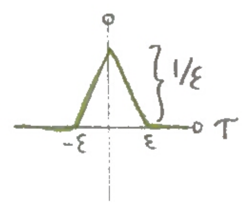
$R_{\Delta_\varepsilon}(t,t+T) = K_{\Delta_\varepsilon}(t,t+T) = \mathrm{Cov}\left(\frac{W(t+\varepsilon)-W(t)}{\varepsilon}, \frac{W(t+T+\varepsilon)-W(t+T)}{\varepsilon}\right) =$

$= \frac{1}{\varepsilon^2}\left(\min(\cancel{t}+\varepsilon, \cancel{t}+T+\varepsilon) - \min(\cancel{t}+\varepsilon, \cancel{t}+T) - \min(\cancel{t}, \cancel{t}+T+\varepsilon) + \min(\cancel{t}, \cancel{t}+T)\right)$

$= \{\text{substract } t\} = \frac{1}{\varepsilon^2}\left(\min(\varepsilon, T+\varepsilon) - m(\varepsilon, T) - \min(0, T+\varepsilon) + \min(0,T)\right)$

So it doesn't depend on t.
We have four options now: $T<-\varepsilon$, $-\varepsilon \le T \le 0$, $0 \le T \le \varepsilon$, $T > \varepsilon$.

$$\left.\begin{array}{ll} 1/\varepsilon^2(T+\varepsilon-T-(T+\varepsilon)+T) = 0 & , T < -\varepsilon \\ 1/\varepsilon^2(T+\varepsilon-T-0+T) = (T+\varepsilon)/\varepsilon^2 & , -\varepsilon \le T \le 0 \\ 1/\varepsilon^2(\varepsilon - T - 0 + 0) = (\varepsilon - T)/\varepsilon^2 & , 0 \le T \le \varepsilon \\ 1/\varepsilon^2(\varepsilon - \varepsilon - 0 + 0) = 0 & , T > \varepsilon \end{array}\right\} \Rightarrow$$

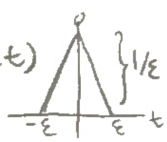


delta (δ) Dirac-distribution $\delta(t) =$ limit as $\varepsilon \to 0$ of any function $f_\varepsilon(t) \ge 0$ s.t

$\int_{-\infty}^{\infty} f_\varepsilon(t)dt = 1$

$f_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ for $t \ne 0$

$\int_{-\infty}^{\infty} f(t)$



$dt \to 0$ as $\varepsilon \to 0$ for f "smooth"